Journal of Global Optimization **6:** 77–86, 1995.

Planar Point-Objective Location Problems with Nonconvex Constraints: A Geometrical Construction

E. CARRIZOSA, E. CONDE, M. MUÑOZ-MARQUEZ and J. PUERTO
Facultad de Matematicas. C/Tarfia s/n. Universidad de Sevilla, 41012 Sevilla, Spain, (email: ECARRIZ@OBELIX.CICA.ES)

(Received: 23 September 1993; accepted: 25 August 1994)

Abstract. The planar point-objective location problem has attracted considerable interest among Location Theory researchers. The result has been a number of papers giving properties or algorithms for particular instances of the problem. However, most of these results are only valid when the feasible region where the facility is to be located is the whole space $\mathbb{R}^2$, which is a rather inaccurate approximation in many real world location problems.
In this paper, the feasible region is allowed to be any closed, not necessarily convex, set S in $\mathbb{R}^2$. The special structure of this nonconvex vector-optimization problem is exploited, leading to a geometrical resolution procedure when the feasible region S can be decomposed into a finite number of (not necessarily disjoint) polyhedra.

Key words: Location, convexity, vector optimization, weak efficiency.

Introduction

Let A be a finite set of points in $\mathbb{R}^2$ (demand points). A facility is to be located at some point x within a feasible set $S \subseteq \mathbb{R}^2$ in such a way that all the demand points have the facility as close as possible (closeness is measured by some metric d in $\mathbb{R}^2$). This leads us to the following vector-optimization problem, known in the literature as the planar point-objective location problem (Wendell and Hurter, 1973):

$$\textbf{POLP}(A, S): \quad \min_{x \in S}(d(x, a) : a \in A),$$

where the minimization must be understood in the multiobjective sense (see, e.g., Chankong-Haimes, 1983).

Among the different solution concepts proposed for **POLP**(A, S) (see, e.g., Wendell and Hurter, 1973), in this paper we address the problem of determining the set $WE(A, S)$ of *(globally) weakly efficient* solutions to **POLP**(A, S).

We recall that a point $x \in S$ is said to be a weakly efficient solution to **POLP**(A, S) (respect. *locally efficient solution*) iff there exists no $y \in S$ such that $d(y, a) < d(x, a)$ for all $a \in A$ (respect. there exists some V, open neighborhood of x in S, such that no $y \in V \cap S$ verifies $d(y, a) < d(x, a)\ \forall a \in A$).

Several reasons motivate the study of the aforementioned solution set. First of all, in real problems planners might have to decide the location of the facility according to different and conflicting criteria. In such cases, the knowledge of

$WE(A, S)$ could help them to decide where to locate (or at least where not to locate): the definition above suggests that only the points in $WE(A, S)$ should be considered as admissible locations for a desirable facility. On the other hand, this solution set has been identified as the set of optimal solutions to a wide family of single-objective optimization problems (Plastria, 1984), thus the determination of $WE(A, S)$ provides a broad-sense sensitivity analysis, enabling the statement of localization theorems (Juel and Love, 1983; Plastria, 1984) for single-objective problems.

A number of papers have been focused on finding the set of weakly efficient solutions and other related solution sets (among others, Durier and Michelot, 1986; Pelegrín and Fernández, 1988; Plastria, 1983; Wendell *et al.*, 1977, and some of the references mentioned above). However, in most of the studies we are aware of, the feasible set S is assumed to coincide with $\mathbb{R}^2$, which, as pointed out by Hansen *et al.* (1982), may be a rather inaccurate approximation in real world problems.

In this paper we study the constrained problem **POLP**(A, S) when d is the Euclidean distance, and the feasible region S is an arbitrary closed set in $\mathbb{R}^2$. In Section 1, we review some recent results that characterize the set $WE(A, S)$ when S is a closed convex set in $\mathbb{R}^2$. In Section 2 the concept of *closed convex decomposition* is introduced, which leads to a characterization of $WE(A, S)$. The properties obtained enable a geometrical construction for $WE(A, S)$ when S is a finite union of polyhedra in $\mathbb{R}^2$. Such construction is discussed in Section 3.

1. Notation and Basic Results

Throughout this paper, the following notation is used:

For any $X \subseteq \mathbb{R}^2$, let bd(X) represent its boundary, i(X) its interior and conv(X) its convex hull.

In a recent paper (Carrizosa *et al.*, 1993), the authors have characterized $WE(A, S)$ when S is a closed convex set in $\mathbb{R}^2$. As some of their results are the cornerstone for the nonconvex case discussed in this paper, they are stated here.

THEOREM 1. *Let X be a nonempty closed convex set in $\mathbb{R}^2$, and let $y \in \mathbb{R}^2$. The following statements are equivalent:*

(i) There exists no $x \in X$ such that

$$d(a, x) < d(a, y) \quad \text{for all } a \in A.$$

(ii) There exists $a^* \in conv(A)$ such that

$$d(a^*, y) \leqslant d(a^*, x) \quad \text{for all } x \in X.$$

Theorem 1 is used in Carrizosa *et al.* (1993) to characterize the set $WE(A, S)$ when S is closed and convex.

Given a nonempty closed convex set X in $\mathbb{R}^2$ and $y \in \mathbb{R}^2$, denote by $\text{proj}_X(y)$ the point in X closest to y, i.e.:

$$\text{proj}_X(y) = \arg\min_{x \in X} d(x, y).$$

Besides, for any $Y \subseteq \mathbb{R}^2$, we denote by $\text{proj}_X(Y)$ the set

$$\text{proj}_X(Y) = \bigcup_{y \in Y} \text{proj}_X(y),$$

THEOREM 2. *For any nonempty closed convex set S in $\mathbb{R}^2$,*

$$WE(A, S) = \text{proj}_S\, \text{conv}(A).$$

Remark that, as a particular case, taking $S = \mathbb{R}^2$, one reobtains the well-known result $WE(A, \mathbb{R}^2) = \text{conv}(A)$ (e.g. Wendell and Hurter, 1973). See Carrizosa *et al.* (1993) for further implications.

2. Weak Efficiency and ccd's

Simple counterexamples show that, as soon as the convexity assumption on S is dropped, Theorem 2 fails.

Besides, the equality between *local* and *global* weak efficiency does not hold any longer. However, not everything is lost: although Theorem 2 requires the feasible region to be convex, it can be used for nonconvex situations by splitting S into convex pieces:

DEFINITION 1. Let S be a nonempty set in $\mathbb{R}^2$. The family of sets $\{S_i : i \in I\}$ is said to be a *closed convex decomposition* (ccd) of S iff each S_i is a nonempty closed convex set, and $S = \cup_{i \in I} S_i$.

It is easy to see that, if $\{S_i : i \in I\}$ is a ccd of S, then (by Theorem 2) $WE(A, S) \subseteq \cup_{i \in I} \text{proj}_{S_i}(\text{conv}(A))$, but equality does not necessarily hold. Anyway, as shown in this section, the set $WE(A, S)$ can be characterized in terms of its ccd's.

First we introduce some notation: For any $x \in \mathbb{R}^2$ and $Y \subseteq \mathbb{R}^2$, $Y \neq \emptyset$, let $B(x; Y) = \{z \in \mathbb{R}^2 : d(x, z) \leqslant d(x, y)\, \forall y \in Y\}$; furthermore, for any nonempty set $X \subseteq \mathbb{R}^2$, let $B(X; Y) = \cup_{x \in X} B(x; Y)$.

An example is depicted in Figure 1, where the set X is the closed segment with endpoints a and b, and Y is the shaded triangle.

The next theorem shows that the problem of finding $WE(A, S)$ can always be reduced to the determination of (intersections of) sets of the form $B(X; Y)$.

THEOREM 3. *Let S be an arbitrary set in $\mathbb{R}^2$, and let $\{S_i : i \in I\}$ be a ccd of S. Then*

$$WE(A, S) = \bigcap_{i \in I} B(\text{conv}(A); S_i) \cap S.$$

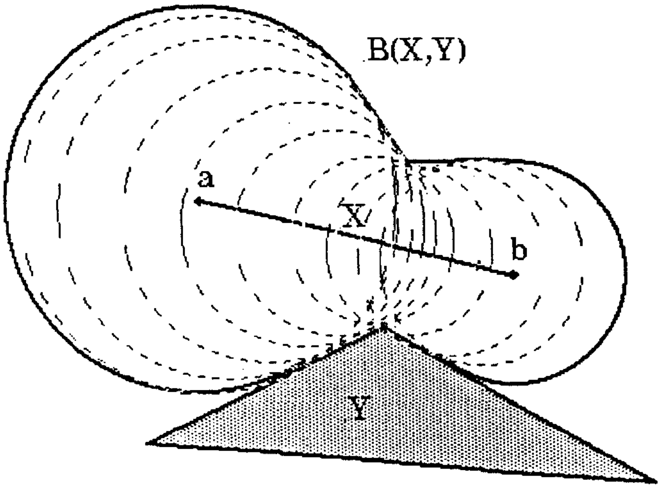

Fig. 1. An example of $B(X, Y)$.

Proof. Let $x \in S = \cup_{i \in I} S_i$. By definition, we have: $x \in WE(A, \cup_{i \in I} S_i)$ iff the set $\{y \in S_i : d(y, a) < d(x, a)\ \forall a \in A\}$ is empty for all $i \in I$. By Theorem 1, the latter assertion is equivalent to

$$\forall i \in I\ \ \exists a_i \in \text{conv}(A) \text{ such that } d(a_i, x) \leqslant \min_{y \in S_i} d(a_i, y)$$

i.e.

$$\forall i \in I\ \ \exists a_i \in \text{conv}(A) \text{ such that } x \in B(a_i; S_i)$$

i.e. (recall that $B(\text{conv}(A); S_i) = \cup_{a \in \text{conv}(A)} B(a; S_i)$)

$$x \in B(\text{conv}(A); S_i) \text{ for all } i \in I$$

i.e.

$$x \in \bigcap_{i \in I} B(\text{conv}(A); S_i).$$

Thus $WE(A, S) = (\cap_{i \in I} B(\text{conv}(A); S_i)) \cap S$, as asserted.

COROLLARY 1. *The set of weakly efficient points does not change if all the points in A which are not extreme points of conv(A) are deleted from A.*

Observe that the ccd for a set is not necessarily unique, so that the simpler the ccd, the easier the determination of $WE(A, S)$. For practical purposes, it would be of interest to simplify the expression obtained in Theorem 3. The definition below gives some help in this sense.

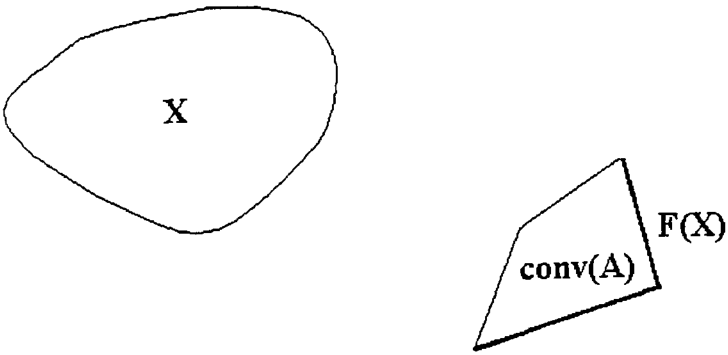

Fig. 2. conv(A)-furthest points from X.

DEFINITION 2. Let X, Y be two nonempty sets in $\mathbb{R}^2$, X being convex and closed. A point $y \in Y$ is said to be Y-furthest from X iff $\{z \in Y : y \in [z, \text{proj}_X(z)]\} = \{y\}$.

For simplicity, the set of conv(A)-furthest points from X is denoted hereafter $F(X)$. See Figure 2 for an illustration.

THEOREM 4. *For any nonempty closed convex set X in $\mathbb{R}^2$, $B(conv(A); X) = (conv(A) \cap X) \cup B(F(X)\backslash X; X)$.*

Proof. By definition of $B(\cdot;\cdot)$ and $F(X)$, and using the fact that $B(a; X) = \{a\}$ whenever $a \in X$, we only have to show that

$$B(\text{conv}(A); X)\backslash(\text{conv}(A) \cap X) \subseteq B(F(X)\backslash X; X).$$

Let x be an arbitrary point in $B(\text{conv}(A); X)$, $x \notin \text{conv}(A) \cap X$. Then, there exists $a \in \text{conv}(A)$ such that $x \in B(a; X)$. Furthermore, $a \notin X$; indeed, else $x \in B(a; X) = \{a\}$, thus $x \in \text{conv}(A) \cap X$. If $a \in F(X)\backslash X$, there is nothing to show. Else, $a \notin F(X)$, thus by definition of $F(X)$, there exists $b \neq a$, $b \in \text{conv}(A) \cap (F(X)\backslash X)$ such that $a \in [b, \text{proj}_X(b)]$, thus $x \in B(b; x) \subset B(F(X)\backslash X; X)$, and this completes the proof.

REMARK 3. For practical purposes, some other expressions might be of more interest. For example, in the next section we use $B(\text{conv}(A); X) = (\text{conv}(A) \cap X) \cup B(F(X)\backslash i(X); X)$, whose validity is a consequence of Theorem 4 above.

3. A Geometrical Construction

The general results obtained in Section 2 enable an explicit construction of $WE(A, S)$ for problems with special structure. In this section we address problem **POLP**(A, S) under the assumption that S admits a *finite polyhedral* ccd $\{S_i : 1 \leqslant i \leqslant p\}$, i.e. $\{S_i : 1 \leqslant i \leqslant p\}$ is a ccd of S and each S_i is a polyhedron in $\mathbb{R}^2$.

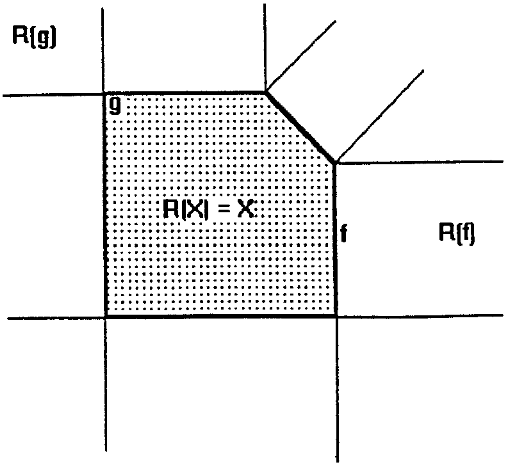

Fig. 3. The planar subdivision $\{R(h) : h \in G\}$.

By Theorem 3, $WE(A, S) = \cap_{i=1}^{p} B(\text{conv}(A); S_i) \cap S$, thus we only need to know how to obtain sets of the form $B(\text{conv}(A); X)$, when X is a polyhedron in $\mathbb{R}^2$.

We only consider here the case when X is a solid polyhedron (i.e., a polyhedron with nonempty interior). The case $i(X) = \emptyset$ is completely analogous, and will not be considered here.

Let G be the set of the faces of X. Recall (see, e.g., Brondsted, 1983), that X is the unique face with dimension 2, the nondegenerate edges of X are the faces with dimension 1, and the vertices of X are the faces with dimension 0.

For any $f \in G$ let $R(f)$ be the polytope

$$R(f) = \text{cl}\{z \in \mathbb{R}^2 : \text{proj}_X(z) \in \text{ri}(f)\}$$

where cl$(\cdot)$ denotes the closure operator, and ri(f) is the relative interior of f. Clearly, $\{R(f) : f \in G\}$ is a polyhedral planar subdivision with the following property:

- If $\dim f = 2$ (i.e., $f = X$), then $R(f) = X$.
- If $\dim f = 1$, then $R(f)$ is an unbounded rectangular region which has f as base.
- If $\dim f = 0$, then $R(f)$ is a cone with vertex at f.

See Figure 3.

By Remark 3, $B(\text{conv}(A); X) = (\text{conv}(A) \cap X) \cup B(F(X) \backslash i(X); X)$, thus the problem is reduced to obtaining $B(F(X) \backslash i(X); X)$.

A further reduction is gotten by observing that the set $F(X) \backslash i(X)$ can be decomposed into a finite number of closed segments (they all in bd (conv(A))), each of them being contained in one of the regions $R(f)$. In fact, the determination

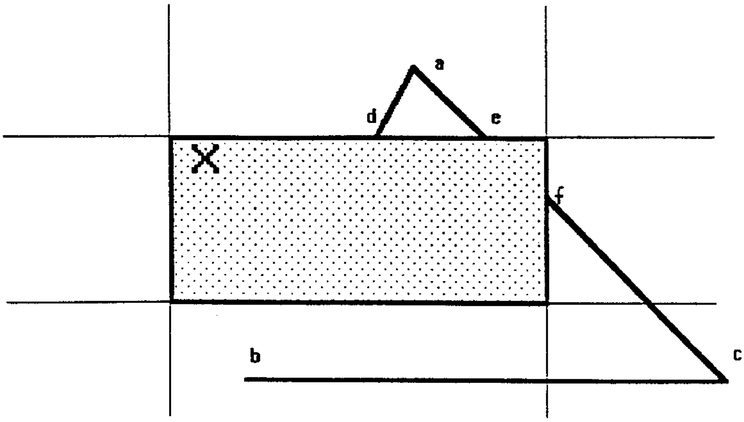

Fig. 4. $F(X)\backslash i(X)$.

of $F(X)$ is straightforward once $\{R(f) : f \in G\}$ is known. This idea is depicted in Figure 4; X is the shaded rectangle, and $A = \{a, b, c\}$. Indeed,

$$F(X)\backslash i(X) = [a, d] \cup [a, e] \cup [f, c] \cup [c, b].$$

Hence, one only needs to know how to determine sets of the form $B(l; X)$, when l is a nondegenerate closed segment, $l = [a, b]$, say, with $l \cap i(X) = \emptyset$, and $l \subseteq R(f)$ for some face f of X, $0 \leqslant \dim f \leqslant 1$. We study separately these two cases:

Case 1. $\dim f = 0$. Then f is a vertex v of X. It is easy to see (by Theorem 1) that $B(l; X) = B(\{a, b\}; \{v\})$, i.e. $B(l; X)$ is the union of the balls centered at a and b and containing v in their boundary.

Case 2. $\dim f = 1$. Denote by r the line containing f, and let $\sigma(\cdot)$ be the symmetry with respect to the line containing l. Then, by definition, $B(l; X)$ is the set of discs whose center is a point in $l = [a, b]$ and are tangent to r (thus also tangent to $\sigma(r)$). Denoting respectively by a' and b' the points $\text{proj}_X(a)$ and $\text{proj}_X(b)$, it follows that $B(l; X) = B(a; \{a'\}) \cup B(b; \{b'\}) \cup \text{conv}(\{a', b', \sigma(a'), \sigma(b')\}$. An example is depicted in Figure 5.

To conclude this section, we summarize the procedure of determination of $WE(S)$ when $S = \cup_{k=1}^{p} S_k$ and each S_k is a polyhedron in $\mathbb{R}^2$.

- For any k, $1 \leqslant k \leqslant p$ do
- Find G_k, the set of faces of S_k
- Find $\{R(f) : f \in G_k\}$
- Decompose $F(S_k)\backslash i(S_k)$ into a finite family L_k of closed segments, each of which is wholly contained in some $R(f)$.

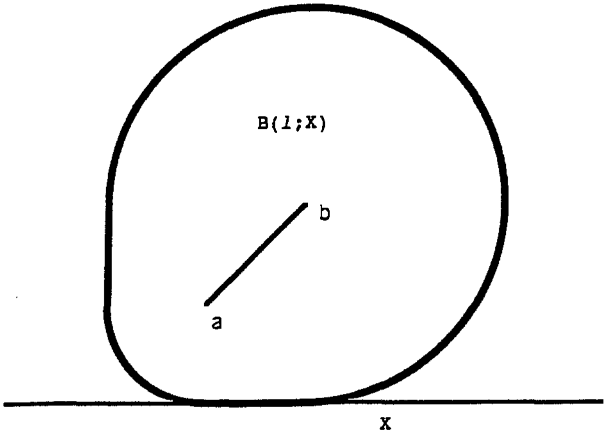

Fig. 5. $B(l; X)$.

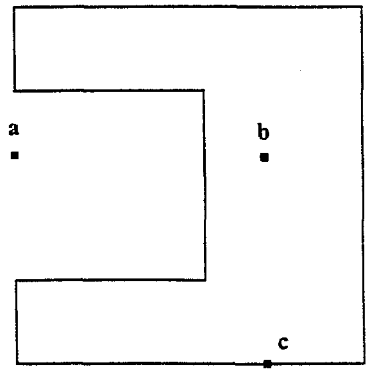

Fig. 6. (a) The example.

· For any $l \in L_k$ find $B(l; S_k)$
· $B(\mathrm{conv}(A); S_k) \leftarrow \cup_{l \in L_k} B(l; S_k)$
· $WE(S) \leftarrow \{\mathrm{conv}(A) \cup (\cap_{k=1}^{p} B(\mathrm{conv}(A); S_k))\} \cap S$.

As an illustration consider the example depicted in Figure 6: $A = \{a, b, c\}$, and the feasible set S is the polygonal region represented in Figure 6(a).

A ccd for S is $\{S_1, S_2, S_3\}$ (see Figure 6(b)), from which it is easily seen that the set $WE(A, S)$ consists of the thick lines and the shadowed area in Figure 6(c), where the point d is at distance $d(a, \mathrm{proj}_{S_2}(a))$ from a.

4. Concluding Remarks

In this paper we have addressed the point-objective location problem under general (not necessarily convex) locational constraints. The concept of ccd's is introduced,

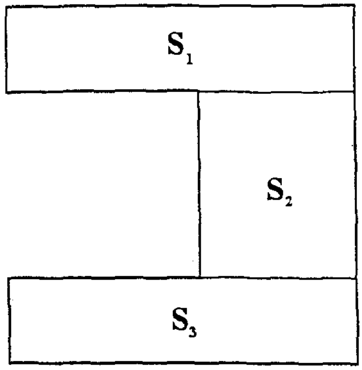

Fig. 6. (b) The ccd S_1, S_2, S_3.

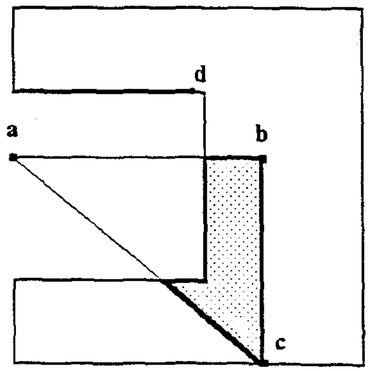

Fig. 6. (c) $WE(A, S)$.

which enables the characterization of the set $WE(A, S)$ of weakly efficient solutions.

When S admits a finite polyhedral ccd, $WE(A, S)$ can be effectively constructed. Furthermore, the geometrical construction can be adapted to arbitrary ccd's $\{S_i : i \in I\}$ as soon as the corresponding sets $B(l; S_i)$ can be explicitly determined.

Such is the case, for example, of discs: it can be seen that, if X is a disc, $B(l; X)$ is a region enclosed by some arcs of circumference, which can be readily obtained.

References

Brondsted, A. (1983), *An Introduction to Convex Polytopes*, Springer-Verlag, New York.

Carrizosa, E., Conde, E., Fernández, F., and Puerto, J. (1993), A Geometrical Characterization of Efficient Points in Constrained Euclidean Location Problems, *Operations Research Letters* **14**, 291–295.

Chankong, V. and Haimes, Y.Y. (1983), *Multiobjective Decision Making*, North-Holland, New York.

Durier, R. and Michelot, C. (1986), Sets of Efficient Points in a Normed Space, *Journal of Mathematical Analysis and Applications* **117**, 506–528.

Hansen, P., Peeters, D., and Thisse, J.F. (1982), An Algorithm for a Constrained Weber Problem, *Management Science* **28**, 1285–1295.
Juel, H. and Love, R.F. (1983), Hull Properties in Location Problems, *European Journal of Operational Research* **12**, 262–265.
Pelegrín, B. and Fernández, F.R. (1988), Determination of Efficient Points in Multiple-Objective Location Problems, *Naval Research Logistic* **35**, 697–705.
Plastria, F. (1983), *Continuous location problems and cutting plane algorithms*. Ph.D. dissertation, Vrije Universiteit Brussel. Brussels.
Plastria, F. (1984), Localization in Single Facility Location, *European Journal of Operational Research* **18**, 215–219.
Wendell, R.E. and Hurter, A.P. (1973), Location Theory, Dominance and Convexity, *Operations Research* **21**, 314–320.
Wendell, R.E., Hurter, A.P., and Lowe, T.J. (1977), Efficient Points in Location Problems, *AIEE Trans.* **9**, 238–246.